Heavy Quarks in Precision Measurements at the LHC

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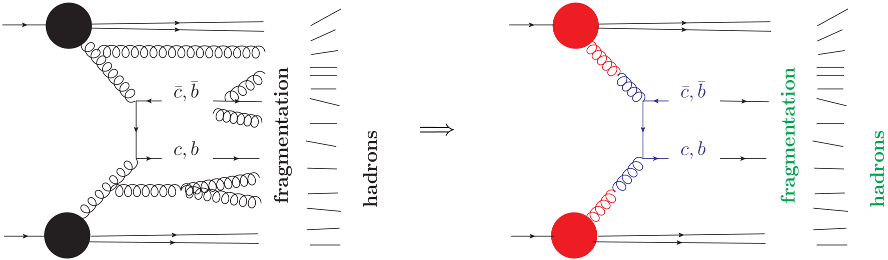
Introduction: Factorization – the general picture



Factorization formula (schematically) for hadronic process

$$A + B \rightarrow H + X : \quad d\sigma_H = f_{A\rightarrow a}(x_1) f_{B\rightarrow b}(x_2) \otimes d\sigma_{ab\rightarrow Q\bar{Q}} \otimes D_{Q\rightarrow H}(z)$$

$d\sigma_H$ = hadronic observable, e.g., in p_T etc;
$d\sigma_Q$ = partonic scattering cross section;
$f_{A/B\rightarrow a/b}$ = non–perturbative parton distribution function; the probability to find parton a(b) in hadron A(B)
$D_{Q\rightarrow H}$ = non–perturbative fragmentation function

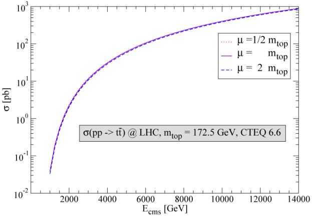
Top quark life–time of $\propto 5 \times 10^{-25} s$ is too short for them to fragment into hadrons

Top quark production

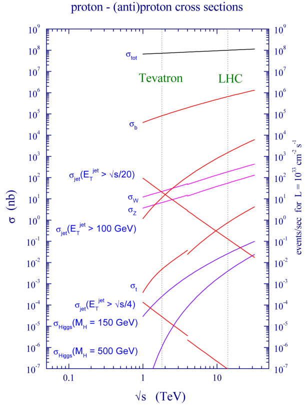
The high energy scale at the LHC demands the inclusion of heavy quarks into calculations: We are in an area where m^2/Q^2–effects are of order $O(1)$ and thus not negligible.

Additionally, top quarks are produced for the first time in large amounts:

- The high energy of the LHC implies that more parton pairs have enough energy to produce a top quark pair
- The LHC luminosity is much higher than that for example of the Tevatron



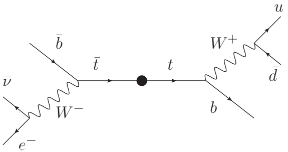
Moch, Uwer, Langenfeld (2009)



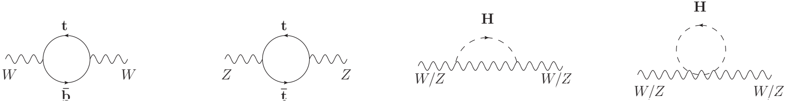
Campbell, Huston, Stirling (2006)

Top quark life–time of $\propto 5 \times 10^{-25}s$ is too short for them to fragment into hadrons;

they can be identified by their decay products into Ws and b-jets:



- Background process: Discovering BSM particles in $t\bar{t}$–channel:
  High–energy jets, leptons and missing transverse energy are typical signals for SUSY as well as for $t\bar{t}$–decays $\Rightarrow$ Undiscovered particles could decay to top quark pairs leading to an increase of the measured cross section
- Precise measurement of the top–quark mass: One ingredient as contribution to the W boson mass $\rightarrow$ consistency check of the SM Higgs boson mass



$\rightarrow$ *See also the talks in the top quark sessions at this conference for many more details*

Top quarks at NNLO

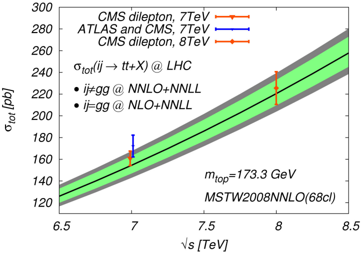
For precise statements about top quarks, one needs the theoretical input in terms of perturbative calculations to higher orders.

Status of the calculations:
Many contributions for approximate calculations from various groups (and subgroups):
[Beneke, Falgari, Klein, Piclum, Schwinn, Ubiali, Yan; Ahrens, Ferroglia, Neubert, Pecjak, Yang; Kidonakis; Aliev, Lacker, Langenfeld, Moch, Uwer, Vogt, Wiedermann; Cacciari, Czakon, Mangano, Mitov, Nason...]

First genuine NNLO QCD corrections to top-pair production at hadron colliders for all fermionic and quark-gluon reaction [Bärnreuther, Czakon, Mitov (2012); Czakon, Mitov (2012)]
gluon-gluon to come.

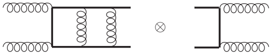
Top++: Computer program for the calculation of the total top–pair cross–section at hadron colliders

Contributions from many people...



Contributing diagrams for the calculation:

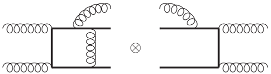
Neglecting the details, one needs in principle:



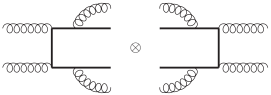
$q\bar{q} \rightarrow Q\bar{Q}$: numerically: Czakon (2007); partly confirmed by analytic results: Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008/09)

$gg \rightarrow Q\bar{Q}$: in preparation (numerically): Czakon, Bärnreuther
Leading color analytically: Bonciani, Ferroglia, Gehrmann, v. Manteuffel, Studerus (2010)

Kniehl, Körner, Merebashvili, Rogal (2005 - 08);
Anastasiou, Aybat (2008)



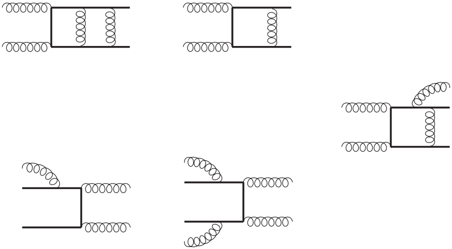
Dittmaier, Uwer, Weinzierl (2007)



Czakon (2010/11),
Anastasiou, Herzog, Lazopoulos (2011)
Abelof, Gehrmann–De Ridder (2011)

One of the major difficulties:

The divergences



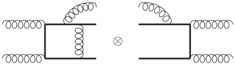
Each loop contributes an ultraviolet divergence $\frac{1}{\varepsilon}$ "plus" possible $\frac{1}{\varepsilon}$ from soft and collinear infrared divergences $\rightarrow$ complicated divergence structure possibly up to divergences $\frac{1}{\varepsilon^4}$.

One has to handle these divergences, for example the soft divergences, in intermediate steps of numerical implementations of the amplitudes!

My own little contribution...

I.B., Czakon, Mitov (2011)

Integrate over real external gluons, e.g.:



$\rightarrow$ this can generate a divergence when the gluon is soft or collinear to the external legs.

The general idea:

Construct a subtraction term in the soft/collinear limit for the squared amplitude.

$$\sigma_{q\bar{q}\to t\bar{t}} = \int_{d\Phi_4} \left(d\sigma^{RR} - d\sigma^{S_{RR}} \right) + \int_{d\Phi_3} \left(d\sigma^{RV} - d\sigma^{S_{RV}} \right)$$

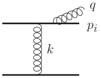
$$+ \int_{d\Phi_2} \left(d\sigma^{VV} + d\sigma^{SV} \right) + \int_{d\Phi_4} d\sigma^{S_{RR}} + \int_{d\Phi_3} d\sigma^{S_{RV}}$$

Where the subtraction terms $d\sigma^{S_{RR}}$ and $d\sigma^{S_{RV}}$ mimic the singular structure of the double–real and the 1–loop 1–real contributions respectively, and can be integrated analytically.

Following an idea for massless external fermions by Catani, Grazzini (2000):

Soft limit: External gluon becomes soft: $q \to \lambda q, \lambda \to 0$;

At the tree level in eikonal approximation:



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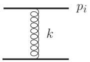


$\times\ J^0(q)$

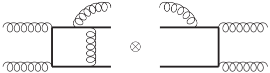
$$|M_a^{(0)}(n+1;q)\rangle = g_s\,\mu^\epsilon J^{(0)}(q)|M^{(0)}(n)\rangle$$

Massive and massless:

$$J_a^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q} \equiv \sum_{i=1}^{n} T_i^a e_i^\mu \,,$$

This factorization is not fully working at the one–loop level.

$$|M_a^{(1)}(n+1;q)\rangle = \underbrace{g_s\,\mu^\epsilon J^{(0)}(q)|M^{(1)}(n)\rangle}_{\text{factorizable}} + \underbrace{g_s\,\mu^\epsilon J^{(1)}(q)|M^{(0)}(n)\rangle}_{\text{non-factorizable}}$$



$$\langle M_a^{(0)}(n+1;q)|M_a^{(1)}(n+1;q)\rangle$$
$$= g_s^2\,\mu^{2\epsilon}\langle M^{(0)}(n)|J^{(0)}(q)\,J^{(0)}(q)|M^{(1)}(n)\rangle$$
$$+ g_s^2\,\mu^{2\epsilon}\langle M^{(0)}(n)|J^{(0)}(q)\,J^{(1)}(q)|M^{(0)}(n)\rangle$$

$J^{(0)}(q^2)$ and $J^{(1)}(q^2)$ are universal, process independent and hence, calculated once, can be used in other applications!

The determination of $J^{(1)}(q)$ involves the calculation of increasingly complicated integrals in different kinematic regions of space– and time–like momenta.

$$J_a^{\mu(1)}(q) = if_{abc}\sum_{i\neq j=1}^{n} T_i^b T_j^c \left(e_i^\mu - e_j^\mu\right) g_{ij}^{(1)}(\epsilon, q, p_i, p_j)\,,$$

$$g_{ij}^{(1)}(C1) = R_{ij}^{[C1]} + i\pi I_{ij}^{[C1]} \equiv\; a_S^b \left(\frac{2(p_i\cdot p_j)\mu^2}{2(p_i\cdot q)2(p_j\cdot q)}\right)^\epsilon \sum_{n=-2}^{r} \epsilon^n \left(R_{ij}^{(n)[C1]} + i\pi I_{ij}^{(n)[C1]}\right)$$

$$R_{ij}^{(-2)[C3]} = -\frac{1}{2}\,,$$

$$R_{ij}^{(-1)[C3]} = \frac{1}{2}\left(-1+\frac{1}{v}\right)\ln(x)+\frac{1}{2}\ln(1+x^2)\,,$$

$$R_{ij}^{(0)[C3]} = \frac{1}{2v}\mathrm{Li}_2(x^2)+\pi^2\left(\frac{19}{24}-\frac{7}{12v}\right)+\frac{1}{v}\ln(v)\ln(x)+\frac{1}{2}\left(1+\frac{1}{v}\right)\ln(x)\ln(1+x^2)$$
$$-\frac{1}{4}\ln^2(1+x^2)+\frac{1}{Q_S}\Big[\left(m_j^2(p_i\cdot q)^2+m_i^2(p_j\cdot q)^2\right)\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)$$
$$+4\left(m_j^2(p_i\cdot q)^2+m_i^2(p_j\cdot q)^2\right)\ln^2(x)-4\frac{m_j^2(p_i\cdot q)^2-m_i^2(p_j\cdot q)^2}{v}\ln\left(\frac{\alpha_i}{\alpha_j}\right)\ln(x)\Big]\,.$$

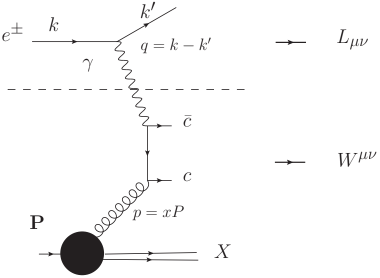
$$R_{ij}^{(1)[C3]} = \frac{1}{v}\Big(-\ln(v)\left(\ln(x)\ln\left(x^2+1\right)+\pi^2\right)+\frac{\ln^3(x)}{12}+\frac{\zeta(3)}{2}$$
$$+\ln(x)\left(\frac{1}{16}\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)+\frac{\mathrm{Li}_2\left(x^2\right)}{2}-\frac{3}{4}\ln^2\left(x^2+1\right)-\frac{5\pi^2}{24}\right)$$
$$-\left(\frac{\mathrm{Li}_2\left(x^2\right)}{2}+\frac{5\pi^2}{12}\right)\ln\left(x^2+1\right)-\frac{1}{2}\left(2\mathrm{Li}_3\left(1-x^2\right)+\mathrm{Li}_3\left(x^2\right)\right)\Big)$$
$$+\frac{1}{Q_S}\Big[(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q)\Big(\frac{32\ln^3(x)}{3}-\frac{280\zeta(3)}{3}-32\ln\left(x^2+1\right)\ln^2(x)$$
$$+\ln\left(x^2+1\right)\left(36\pi^2-4\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)\right)+\left(48\ln^2\left(x^2+1\right)-\frac{40\pi^2}{3}\right)\ln(x)$$
$$-\frac{88}{3}\ln^3\left(x^2+1\right)\Big)$$
$$(m_j^2(p_i\cdot q)^2+m_i^2(p_j\cdot q)^2)\Big(\left(3\ln\left(x^2+1\right)+\ln(x)\right)\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)$$
$$+28\ln^3\left(x^2+1\right)-44\ln^2\left(x^2+1\right)\ln(x)-\frac{70}{3}\pi^2\ln\left(x^2+1\right)+28\ln\left(x^2+1\right)\ln^2(x)$$
$$-\frac{28}{3}\ln^3(x)+\frac{2}{3}\pi^2\ln(x)+\frac{224\zeta(3)}{3}\Big)$$
$$-\frac{(m_j^2(p_i\cdot q)^2-m_i^2(p_j\cdot q)^2)}{v}\ln\left(\frac{\alpha_i}{\alpha_j}\right)\left(4\mathrm{Li}_2\left(x^2\right)+4\ln\left(x^2+1\right)\ln(x)+8\ln(v)\ln(x)-\frac{14\pi^2}{3}\right)$$
$$+(m_i^2(p_j\cdot q)^2+m_j^2(p_i\cdot q)^2-(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q))\Big(\ln^3\left(\frac{\alpha_i}{\alpha_j}\right)-\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)2\ln(v)$$
$$+\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)2(\ln(\alpha_i+v+1)+\ln(-\alpha_j+v+1)+\ln(\alpha_j+v-1)-3\ln(2))$$
$$+\ln\left(\frac{\alpha_i}{\alpha_j}\right)\Big(2\ln^2(v)-12\ln^2(\alpha_i+v+1)+6\ln^2(\alpha_j+v+1)-4\ln^2(x)-12\ln^2\left(x^2+1\right)$$
$$+28\ln(2)\ln(\alpha_i+v+1)+4\ln(v)(\ln(\alpha_i+v+1)-2\ln(\alpha_j+v+1)+\ln(2))-10\ln^2(2)$$
$$-4\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)-8\ln(2)\ln(\alpha_j+v+1)$$
$$+8(2\ln(\alpha_i+v+1)-\ln(-\alpha_j+v+1)+\ln(\alpha_j+v-1)+\ln(\alpha_j+v+1)-3\ln(2))\ln(x)$$
$$+24(\ln(2)-\ln(\alpha_i+v+1))\ln\left(x^2+1\right)+24\ln(x)\ln\left(x^2+1\right)-\frac{2}{3}\pi^2\Big)$$
$$+\frac{32}{3}\ln^3(\alpha_i+v+1)+12\ln^3(\alpha_j+v+1)-32\ln(2)\ln^2(\alpha_i+v+1)-8\ln(v)\ln^2(\alpha_j+v+1)$$
$$+4\ln(\alpha_i+v+1)\ln^2(\alpha_j+v+1)-40\ln(2)\ln^2(\alpha_j+v+1)-8\ln(v)\ln^2(x)-\frac{4}{3}\pi^2\ln(v)$$
$$+8(3\ln(\alpha_i+v+1)+\ln(-\alpha_j+v+1)+\ln(\alpha_j+v-1)-5\ln(2))\ln^2(x)$$
$$+40(\ln(\alpha_i+v+1)+\ln(\alpha_j+v+1)-2\ln(2))\ln^2\left(x^2+1\right)+36\ln^2(2)\ln(\alpha_i+v+1)$$
$$+8\ln(v)\ln(\alpha_i+v+1)\ln\left(\frac{1}{2}(\alpha_j+v+1)\right)-4\ln^2(v)(\ln(\alpha_i+v+1)-\ln(\alpha_j+v+1))$$
$$+\frac{4}{3}\pi^2(-3\ln(\alpha_i+v+1)-4\ln(\alpha_j+v+1)+7\ln(2))+8\ln(2)\ln(v)\ln(\alpha_j+v+1)$$
$$-8\ln(2)\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)+44\ln^2(2)\ln(\alpha_j+v+1)+4\ln^2(v)\ln(x)$$
$$-24\ln^2(\alpha_i+v+1)\ln(x)-4\ln^2(\alpha_j+v+1)\ln(x)+8\ln(v)(\ln(2)-\ln(\alpha_i+v+1))\ln(x)$$
$$+56\ln(2)\ln(\alpha_i+v+1)\ln(x)-8\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)\ln(x)-36\ln^2(2)\ln(x)$$
$$+16\ln(2)\ln(\alpha_j+v+1)\ln(x)+32\ln^2(\alpha_i+v+1)\ln\left(x^2+1\right)+80\ln^2(2)\ln\left(x^2+1\right)$$
$$+32\ln^2(\alpha_j+v+1)\ln\left(x^2+1\right)-80\ln(2)\ln(\alpha_i+v+1)\ln\left(x^2+1\right)$$
$$+16\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)\ln\left(x^2+1\right)-80\ln(2)\ln(\alpha_j+v+1)\ln\left(x^2+1\right)$$
$$+16(-4\ln(\alpha_i+v+1)-\ln(\alpha_j+v+1)+5\ln(2))\ln(x)\ln\left(x^2+1\right)-\frac{80\ln^3(2)}{3}$$
$$+\left(8\ln\left(\frac{\alpha_i}{\alpha_j}\right)+16\ln(x)\right)\mathrm{Li}_2\left(\frac{1-v}{\alpha_j}\right)+\left(16\ln(x)-8\ln\left(\frac{\alpha_i}{\alpha_j}\right)\right)\mathrm{Li}_2\left(\frac{\alpha_j}{v+1}\right)$$
$$+\left(4\ln\left(\frac{\alpha_i}{\alpha_j}\right)-8\ln(\alpha_i+v+1)-8\ln(\alpha_j+v+1)+24\ln(x)-16\ln\left(x^2+1\right)+4\ln^4(2)\right)$$
$$\times\left(\mathrm{Li}_2\left(\frac{v-1}{\alpha_j}\right)-\mathrm{Li}_2\left(\frac{\alpha_j}{\alpha_j-v+1}\right)\right)$$
$$+\left(4\ln\left(\frac{\alpha_i}{\alpha_j}\right)-8\ln(\alpha_i+v+1)-8\ln(\alpha_j+v+1)-8\ln(x)-16\ln\left(x^2+1\right)+4\ln^4(2)\right)$$
$$\times\left(\mathrm{Li}_2\left(-\frac{v+1}{\alpha_j}\right)-\mathrm{Li}_2\left(\frac{\alpha_j}{\alpha_j+v+1}\right)\right)$$
$$+8(\ln(v)-\ln(\alpha_j+v+1)+\ln(2))\mathrm{Li}_2\left(-\frac{(v-1)(\alpha_j+v+1)}{(\alpha_j-v+1)(v+1)}\right)-16\ln(x)\mathrm{Li}_2\left(x^2\right)$$
$$-16\mathrm{Li}_3\left(\frac{1-v}{\alpha_j}\right)-16\mathrm{Li}_3\left(\frac{\alpha_j}{\alpha_j-v+1}\right)+8\mathrm{Li}_3\left(\frac{\alpha_j}{v-1}\right)-16\mathrm{Li}_3\left(\frac{v-1}{\alpha_j}\right)+8\mathrm{Li}_3\left(-\frac{\alpha_j}{v+1}\right)$$
$$-8\mathrm{Li}_3\left(-\frac{2v}{\alpha_j-v+1}\right)-16\mathrm{Li}_3\left(\frac{\alpha_j}{v+1}\right)+8\mathrm{Li}_3\left(x^2\right)-16\mathrm{Li}_3\left(-\frac{v+1}{\alpha_j}\right)$$
$$-16\mathrm{Li}_3\left(\frac{v-1}{-\alpha_j+v-1}\right)-16\mathrm{Li}_3\left(\frac{\alpha_j}{\alpha_j+v+1}\right)-8\mathrm{Li}_3\left(\frac{2v}{\alpha_j+v+1}\right)$$
$$-8\mathrm{Li}_3\left(\frac{2\alpha_jv}{(v-1)(\alpha_j+v+1)}\right)-16\mathrm{Li}_3\left(\frac{v+1}{\alpha_j+v+1}\right)+8F_c\left(\frac{\alpha_j}{\alpha_j-v+1},\frac{\alpha_j}{\alpha_j+v+1}\right)\Big)\Big]\,.$$

- Higher order calculations with massive quarks are important to improve the precision on scattering cross sections including top quarks.
  The massive one–loop soft–gluon current was one important ingredient to the NNLO top–quark production calculation of Czakon et al.
  $\rightarrow$ on the way towards percent precision on heavy quark calculations

- Massive higher order calculations are also an important input for other processes, e.g.: Heavy quarks in DIS
  There, these calculations can be used to extract properties of the heavy quarks with high precision $\rightarrow$ "re-use" them as input for the LHC

Higher order calculations to other processes - Heavy quarks in DIS

Assume only light partons in the proton. Heavy quarks (c or b) emerge in final states through hard scattering processes (top outside the HERA region).

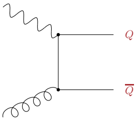


$$Q^2 := -q^2, \quad x := \frac{Q^2}{2p \cdot q}, \quad \text{Bjorken–x}$$

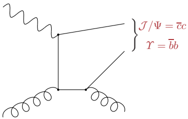
$$\nu := \frac{P \cdot q}{M},$$

$$\frac{d\sigma}{dQ^2\, dx} \sim L^{\mu\nu} W_{\mu\nu}$$

- open c(b) production: $D_u = \overline{u}c$, ... $B_u = \overline{u}b$, ...



- heavy quark resonances: $\overline{c}c = \mathcal{J}/\Psi$ $\overline{b}b = \Upsilon$.



The hadronic tensor cannot be calculated perturbatively.
However, it can be decomposed into several scalar structure functions.

For unpolarized DIS via single photon exchange, the hadronic tensor is given by:

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

In Bjorken limit, $\{Q^2,\ \nu\} \to \infty$, x fixed, at twist $\tau = 2$–level:

$$\underbrace{F_i(x,Q^2)}_{\text{structure functions}} = \sum_j \underbrace{\mathcal{C}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{parton densities, non-perturbative}},$$

$$[f \otimes g](x) = \int_0^1 dx_1 \int_0^1 dx_2 \;\; \delta(x - x_1 x_2)\; f(x_1) g(x_2)$$

Wilson coefficients contain both light and heavy flavor contributions:

$$\mathcal{C}_{i,j}\left(x,\frac{Q^2}{\mu^2},\frac{m_k^2}{\mu^2}\right)=C_{i,j}^{\rm light}\left(x,\frac{Q^2}{\mu^2}\right)+\mathrm{H}_{i,j}\left(x,\frac{Q^2}{\mu^2},\frac{m_k^2}{\mu^2}\right)\ ,k=c,b\ .$$

$$\mathrm{H}_{(2,L),i}^{\rm S}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)=\underbrace{H_{(2,L),i}^{\rm S}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)}_{\gamma+q_{heavy}\to X}+\underbrace{L_{(2,L),i}^{\rm S,NS}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)}_{\gamma+q_{light}\to X}$$

LO contribution to $F_{(2,L)}$ from heavy quark production: photon-gluon fusion

$$F_{(2,L)}^{Q\overline{Q}}(x,Q^2)=4e_c^2a_s\int_{ax}^1\frac{dz}{z}H_{(2,L),g}^{(1)}\left(\frac{x}{z},\frac{m^2}{Q^2}\right)G(z,Q^2)\ ,\quad a=1+4m^2/Q^2\ .$$

General factorization for $F_2^{Q\overline{Q}}(x, Q^2)$ at the level of twist $\tau = 2$:

$$
\begin{aligned}
& F_{2,Q}(x,Q^2,n_f,m) \\
= & \sum_{k=1}^{n_f} e_k^2 \Bigg\{ L_{2,q}^{\mathsf{NS}}\left(n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,n_f) + f_{\overline{k}}(x,\mu^2,n_f)\right] \\
& + \tilde{L}_{2,q}^{\mathsf{PS}}\left(n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,n_f) + \tilde{L}_{2,g}^{\mathsf{S}}\left(n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,n_f) \Bigg\} \\
& + e_Q^2 \left[H_{2,q}^{\mathsf{PS}}\left(n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,n_f) + H_{2,g}^{\mathsf{S}}\left(n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,n_f) \right],
\end{aligned}
$$

with the flavour singlet distribution: $\Sigma(x, \mu^2) = \sum_{k=1}^{n_f} \left[f_k(x, \mu^2) + f_{\overline{k}}(x, \mu^2)\right]$

$F_2^{c\bar{c}}(x, Q^2) \sim 20 - 40\,\%$ of $F_2(x, Q^2)$ for small values of x, but obey different scaling violations.

Unfortunately, the analytic calculation of the full structure functions $F_{2/L}^{Q\bar{Q}}$ is highly (too) complicated. It is only available numerically up to a certain extend in the form of a computer program.

Is there anything we can do on the analytic side?

Yes, we *can* calculate the expression in the limit of $Q^2 \gg m^2$ even up to the three–loop level in Mellin space by now → [I.B., Klein, Blümlein (2009)].

Limit of high Q^2

- In the limit $Q^2 \gg m_h^2$: massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections $(m^2/Q^2)^k$, $k \geq 1$, calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H^{\mathrm{S}}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\mathrm{S}}_{ki}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\mathrm{S}}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton Wilson coefficients}}.$$

- Similar formula for $L^{S,NS}_{(2,L),i}$. Holds for polarized and unpolarized case.
- OMEs obey expansion

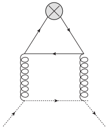
$$A^{\mathrm{S,NS}}_{ki}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\mathrm{S,NS}}_k|i\rangle = \delta_{ki} + \sum_{l=1}^{\infty} a_s^l A^{\mathrm{S,NS},(l)}_{ki}\left(\frac{m^2}{\mu^2}\right), \quad i = q, g$$

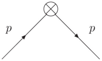
[Buza, Matiounine, Migneron, Smith, van Neerven (1996); Buza, Matiounine, Smith, van Neerven (1997).]

Light flavor Wilson coefficients exist to $O(\alpha_s^3)$: [Moch, Vermaseren, Vogt (2005)]
$\Rightarrow$ one also needs the Heavy Flavour Wilson Coefficients to NNLO to reach the same accuracy.

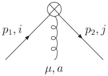
Structure of the OMEs:

The $\otimes$ are operator insertions in light-cone expansion from the photon coupling to the quarks:

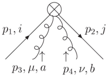




$$\not{\Delta}(\Delta \cdot p)^{N-1}\,, \quad N \geq 1$$



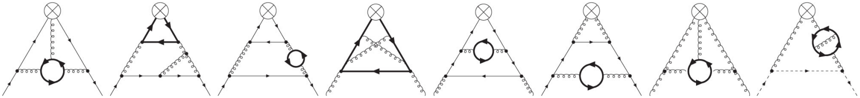
$$g t^a_{ji} \Delta^\mu \not{\Delta} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\,, \quad N \geq 2$$



$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \sum_{0 \leq j < l}^{N-2} \Big[(\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_4)^{l-j-1} (\Delta p_2)^j (t^a t^b)_{ji} + (\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_3)^{l-j-1} (\Delta p_2)^j (t^b t^a)_{ji} \Big]$$
$$N \geq 3$$

Δ: light-like momentum, $\Delta^2 = 0$.

Diagrams contributing to three-loop order:



Expansion up to $O(a_s^3)$ for $F_{2/L}^{Q\overline{Q}}(x,Q^2)$ in Mellin space reads

$$\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(n_f+1) &= a_s^2\left[A_{qq,Q}^{\mathsf{NS},(2)}(n_f+1)\delta_2+\hat{C}_{q,(2,L)}^{\mathsf{NS},(2)}(n_f)\right] \\
&\quad +a_s^3\left[A_{qq,Q}^{\mathsf{NS},(3)}(n_f+1)\delta_2+A_{qq,Q}^{\mathsf{NS},(2)}(n_f+1)C_{q,(2,L)}^{\mathsf{NS},(1)}(n_f+1)+\hat{C}_{q,(2,L)}^{\mathsf{NS},(3)}\right] \\
L_{q,(2,L)}^{\mathsf{PS}}(n_f+1) &= a_s^3\left[\tilde{A}_{qq,Q}^{\mathsf{PS},(3)}(n_f+1)\delta_2+A_{gq,Q}^{(2)}(n_f+1)n_f\tilde{C}_{g,(2,L)}^{(1)}(n_f+1)+n_f\hat{\tilde{C}}_{q,(2,L)}^{\mathsf{PS},(3)}(n_f)\right] \\
L_{g,(2,L)}^{\mathsf{S}}(n_f+1) &= a_s^2A_{gg,Q}^{(1)}(n_f+1)n_f\tilde{C}_{g,(2,L)}^{(1)}(n_f+1)+a_s^3\Big[\tilde{A}_{qg,Q}^{(3)}(n_f+1)\delta_2+A_{gg,Q}^{(1)}(n_f)n_f\tilde{C}_{g,(2,L)}^{(2)}(n_f+1) \\
&\quad +A_{gg,Q}^{(2)}(n_f)n_f\tilde{C}_{g,(2,L)}^{(1)}(n_f+1)+A_{Qg}^{(1)}(n_f)n_f\tilde{C}_{q,(2,L)}^{\mathsf{PS},(2)}(n_f+1)+n_f\hat{\tilde{C}}_{g,(2,L)}^{(3)}(n_f)\Big] \\
H_{q,(2,L)}^{\mathsf{PS}}(n_f) &= a_s^2\left[A_{Qq}^{\mathsf{PS},(2)}(n_f+1)\delta_2+\tilde{C}_{q,(2,L)}^{\mathsf{PS},(2)}(n_f+1)\right]+a_s^3\Big[A_{Qq}^{\mathsf{PS},(3)}(n_f+1)\delta_2+\tilde{C}_{q,(2,L)}^{\mathsf{PS},(3)}(n_f+1) \\
&\quad +A_{gq,Q}^{(2)}(n_f+1)\,\tilde{C}_{g,(2,L)}^{(1)}(n_f+1)+A_{Qq}^{\mathsf{PS},(2)}(n_f+1)\,C_{q,(2,L)}^{\mathsf{NS},(1)}(n_f+1)\Big] \\
H_{g,(2,L)}^{\mathsf{S}}(n_f) &= a_s\left[A_{Qg}^{(1)}(n_f+1)\delta_2+\tilde{C}_{g,(2,L)}^{(1)}(n_f+1)\right] \\
&\quad +a_s^2\Big[A_{Qg}^{(2)}(n_f+1)\delta_2+A_{Qg}^{(1)}(n_f+1)\,C_{q,(2,L)}^{\mathsf{NS},(1)}(n_f+1)+A_{gg,Q}^{(1)}(n_f+1)\,\tilde{C}_{g,(2,L)}^{(1)}(n_f+1) \\
&\quad +\tilde{C}_{g,(2,L)}^{(2)}(n_f+1)\Big]+a_s^3\Big[A_{Qg}^{(3)}(n_f+1)\delta_2+A_{Qg}^{(2)}(n_f+1)\,C_{q,(2,L)}^{\mathsf{NS},(1)}(n_f+1) \\
&\quad +A_{gg,Q}^{(2)}(n_f+1)\,\tilde{C}_{g,(2,L)}^{(1)}(n_f+1)+A_{Qg}^{(1)}(n_f+1)\Big[C_{q,(2,L)}^{\mathsf{NS},(2)}(n_f+1) \\
&\quad +\tilde{C}_{q,(2,L)}^{\mathsf{PS},(2)}(n_f+1)\Big]+A_{gg,Q}^{(1)}(n_f+1)\,\tilde{C}_{g,(2,L)}^{(2)}(n_f+1)+\tilde{C}_{g,(2,L)}^{(3)}(n_f+1)\Big]\,.
\end{aligned}$$

n_f–dependence non–trivial: $\hat{f}(n_f)\equiv f(n_f+1)-f(n_f)$, $\tilde{f}(n_f)\equiv f(n_f)/n_f$.

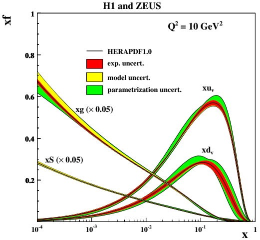
Theory Status of unpolarized Heavy Quark Corrections

- LO: $F_{2,L}(x, Q^2)$ [Witten (1976); Babcock, Sivers (1978); Shifman, Vainshtein, Zakharov (1978); Leveille, Weiler (1979); Glück, Reya (1979); Glück, Hoffmann, Reya (1982).]
- NLO: $F_{2,L}(x, Q^2)$ [Laenen, Riemersma, Smith, van Neerven (1993), (1995)]
  Mellin–space expressions: [Alekhin, Blümlein (2003).].
  asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven (1996); I.B., Blümlein, Klein (2007)]
- NNLO:
  - $F_L(x, Q^2)$ asymptotic: [Blümlein, De Freitas, Klein, van Neerven (2006).]
  - $F_2(x, Q^2)$ asymptotic: in Mellin space for all N apart from genuine constant $O(\alpha_s^3)$ terms. Fixed moments in N for full OMEs [I.B., Blümlein, Klein (2009)]
    Since then, much progress was and is done concerning constant terms:
    [Blümlein, Hasselhuhn, Klein, Schneider (2011); Ablinger et al (2012).]
    with the goal to provide first full Heavy Wilson Coefficients in x space
    [I.B., Blümlein, Klein, Wißbrock, in preparation]
    Even addressing two heavy quark masses: [Ablinger et al (2012).]

The work to NNLO involved the development of new calculational techniques and the use of programs like `SIGMA` by C. Schneider, `MATAD` by M. Steinhauser...
Many interesting things about occurring mathematical objects (harmonic sums...)...

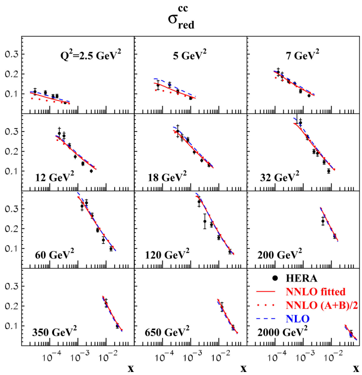
Applications of the OMEs:

OMEs allow the extraction of the gluon density for small x to high precision



Precise determination of charm quark mass from inclusive $F_2(x, Q^2)$

[Alekhin, Blümlein, Daum, Lipka, Moch (2012)]



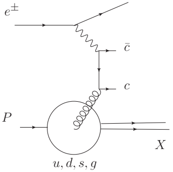
Heavy OMEs also occur as transition functions to define a variable flavor number scheme starting from a fixed flavor number scheme. [Aivazis, Collins, Olness, Tung (1994); Buza, Matiounine, Smith, van Neerven (1998); Chuvakin, Smith, van Neerven (1998).]

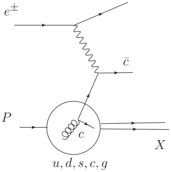
FFNS:

- Fixed order perturbation theory and Fixed number of light partons in the proton.
- The heavy quarks are produced extrinsically only.
- The large logarithmic terms in the heavy quark coefficient functions entirely determine the charm component of the structure function for large values of Q^2.

VFNS:

- Define threshold above which the heavy quark is treated as light, thereby obtaining a parton density.
- Remove the mass singular terms from the asymptotic heavy quark coefficient functions and absorb them into parton densities.
- Heavy Flavor initial state parton densities for the LHC.





The VFNS is derived from the FFNS directly. New parton density appears corresponding to the heavy quark, which is now treated as light (massless).

$\Longrightarrow$ Consistent matching between parton densities for n_f and $n_f + 1$ flavors.
Similarity to the decoupling and matching of α_s from $\alpha_s(n_f) \to \alpha_s(n_f + 1)$

$$\begin{aligned} f_{Q+\bar{Q}}(n_f+1,\mu^2) &= A_{Qq}^{\mathrm{PS}}\left(n_f+1,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{Qg}\left(n_f+1,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\ G(n_f+1,\mu^2) &= A_{gq,Q}\left(n_f+1,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}\left(n_f+1,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \end{aligned}$$

$\Rightarrow$ AB(K)M pdfs [Alekhin, Blümlein, (Klein,) Moch, (2009)].

The schemes and approaches for the treatment of heavy quarks over the full range of the considered scale parameter and the ways of their implementation by different (pdf) groups is an interesting topic on its own...

- Higher order calculations are important in the context of heavy quark contributions in deep–inelastic scattering.

  The calculation of the massive operator matrix elements to higher orders allows the precise determination of the gluon pdf for small x from DIS data, the charm quark mass, etc; they allow the construction of heavy flavour pdfs from initial gluon and light-quark pdfs in a VFNS.

- Massive higher order calculations are important to obtain precise theoretical input in various areas of scattering processes and collider physics at the LHC and possible future (linear) colliders!

  $\Rightarrow$ This includes the demand for an improvement of calculational techniques for (massive) Feynman diagrams to higher orders...